Physical fields and Clifford algebras

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Abstract

The physical fields (electromagnetic and electron fields) considered in the framework of Clifford algebras $\mathbf{C}_2$ and $\mathbf{C}_4$. The electron field described by algebra $\mathbf{C}_4$ which in spinor representation is realized by well-known Dirac γ-matrices, and by force of isomorphism $\mathbf{C}_4 \cong \mathbf{C}_2 \otimes \mathbf{C}_2$ is represented as a tensor product of two photon fields. By means of this introduced a system of electron field equations, which in particular cases is coincide with Dirac's and Maxwell's equations.

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It is well-known that Maxwell's equations

$$\begin{aligned}\mathrm{curl}\mathbf{E}+\dot{\mathbf{H}} &= 0,\\ \mathrm{div}\mathbf{H} &= 0,\\ \mathrm{curl}\mathbf{H}-\dot{\mathbf{E}} &= \mathbf{j},\\ \mathrm{div}\mathbf{E} &= \varrho\end{aligned}$$

and Dirac's equations

$$(i\gamma_\mu\frac{\partial}{\partial x_\mu}-m)\psi=0$$

may be rewriten in spinor form[1]

$$\begin{aligned}\partial^{\lambda\dot{\mu}}f^\rho_\lambda &= 0,\\ \partial^{\lambda\dot{\mu}}f^\rho_\lambda &= s^{\rho\dot{\mu}}\end{aligned}$$

and

$$\begin{aligned}\partial^{\lambda\dot{\mu}}\eta_{\dot{\mu}}+im\xi^\lambda &= 0,\\ \partial_{\lambda\dot{\mu}}\xi^\lambda+im\eta_{\dot{\mu}} &= 0,\end{aligned}$$

where

$$(\partial^{\lambda\dot{\mu}})=\begin{bmatrix}\partial^{1\dot{1}} & \partial^{1\dot{2}}\\ \partial^{2\dot{1}} & \partial^{2\dot{2}}\end{bmatrix}=\begin{bmatrix}\partial_0+\partial_3 & \partial_1+i\partial_2\\ \partial_1-i\partial_2 & \partial_0-\partial_3\end{bmatrix}, \tag{1}$$

$$(f^\rho_\lambda)=\begin{bmatrix}f^1_1 & f^1_2\\ f^2_1 & f^2_2\end{bmatrix}=\begin{bmatrix}F_3 & F_1+iF_2\\ F_1-iF_2 & -F_3\end{bmatrix}- \tag{2}$$

-the symmetric spin-tensor of complex electromagnetic field $\mathbf{F}=\mathbf{E}+i\mathbf{H}$. The quantities $\xi^\lambda=(\xi^1,\xi^2)$ and $\eta_{\dot{\mu}}=(\eta_{\dot{1}},\eta_{\dot{2}})$ are make up a bispinor

$$\psi=\begin{bmatrix}\xi^1\\ \xi^2\\ \eta_{\dot{1}}\\ \eta_{\dot{2}}\end{bmatrix}.$$

The spinors ξ^λ and co-spinors $\eta_{\dot{\mu}}$ are satisfy to the following conditions:

$$\xi_1=\xi^2,\;\; \xi_2=-\xi^1,\;\; \eta^{\dot{1}}=-\eta_{\dot{2}},\;\; \eta^{\dot{2}}=\eta_{\dot{1}}.$$

Besides, the quantities $\xi^\lambda=(\xi^1,\xi^2)$ and $\eta_{\dot{\mu}}=(\eta_{\dot{1}},\eta_{\dot{2}})$ be vectors of two-dimensional complex spaces (spin-spaces $S_2(i)$ and $\dot{S}_2(i)$); the each of these spin-spaces is homeomorphic to extended complex plane. It is well-known that the each spin-space be a space of linear representation of the some Clifford algebra[2], in this case it is algebra $\mathbf{C}_2$ (so-called the algebra of hyperbolic bi-quaternions). The motion group of each spin-spaces $S_2(i)$ and $\dot{S}_2(i)$ is isomorphic to a group $\mathrm{SL}(2;\mathbf{C})$ which be a double-meaning representation of Lorentz

group. By force of basic isomorphism $\mathbf{C}_2 \cong \mathrm{M}_2(\mathbf{C})$ for the linear transformations of spinors of the spaces $S_2(i)$ and $\dot{S}_2(i)$ we have:

$$\begin{bmatrix} \xi^{1'} \\ \xi^{2'} \end{bmatrix} = M \begin{bmatrix} \xi^1 \\ \xi_2 \end{bmatrix},$$

$$\begin{bmatrix} \eta_{\dot{1}}{}' \\ \eta_{\dot{2}}{}' \end{bmatrix} = \dot{M} \begin{bmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix},$$

where $M, \dot{M} \in \mathrm{M}_2(\mathbf{C})$.

Further on, consider a Clifford algebra $\mathbf{R}_3$ over a field of real numbers. The units of this algebra are satisfy to the following conditions: $\mathbf{e}_i^2 = 1$, $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ $(i, j = 1, 2, 3)$.

Let

$$\begin{aligned} \mathcal{A}_0 &= \partial_0\mathbf{e}_0 + \partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \partial_3\mathbf{e}_3, \\ \mathcal{A}_1 &= A_0\mathbf{e}_0 + A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3, \end{aligned} \tag{3}$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ be elements of $\mathbf{R}_3$. The coefficients of these elements be partial derivatives and components of vector-potential, respectively.

Make up now the exterior product of elements (3):

$$\begin{aligned} \mathcal{A}_0\mathcal{A}_1 &= (\partial_0\mathbf{e}_0 + \partial_1\mathbf{e}_1 + \partial_2\mathbf{e}_2 + \partial_3\mathbf{e}_3)(A_0\mathbf{e}_0 + A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3) = \\ &= (\underbrace{\partial_0A_0 + \partial_1A_1 + \partial_2A_2 + \partial_3A_3}_{E_0})\mathbf{e}_0 + (\underbrace{\partial_0A_1 + \partial_1A_0}_{E_1})\mathbf{e}_0\mathbf{e}_1 + \\ &(\underbrace{\partial_0A_2 + \partial_2A_0}_{E_2})\mathbf{e}_0\mathbf{e}_2 + (\underbrace{\partial_0A_3 + \partial_3A_0}_{E_3})\mathbf{e}_0\mathbf{e}_3 + (\underbrace{\partial_2A_3 - \partial_3A_2}_{H_1})\mathbf{e}_2\mathbf{e}_3 + \\ &+ (\underbrace{\partial_3A_1 - \partial_1A_3}_{H_2})\mathbf{e}_3\mathbf{e}_1 + (\underbrace{\partial_1A_2 - \partial_2A_1}_{H_3})\mathbf{e}_1\mathbf{e}_2. \end{aligned} \tag{4}$$

The scalar part $E_0 \equiv 0$, since the first bracket in (4) be a Lorentz condition $\partial_0A_0 + \mathrm{div}\mathbf{A} = 0$. It is easily seen that the other bracket be components of electric and magnetic fields: $-E_i = -(\partial_iA_0 + \partial_0A_i)$, $H_i = (\mathrm{curl}\mathbf{A})_i$.

Since in this case the element $\omega = \mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is belong to a center of $\mathbf{R}_3$, then

$$\omega\mathbf{e}_1 = \mathbf{e}_1\omega = \mathbf{e}_2\mathbf{e}_3, \quad \omega\mathbf{e}_2 = \mathbf{e}_2\omega = \mathbf{e}_3\mathbf{e}_1, \quad \omega\mathbf{e}_3 = \mathbf{e}_3\omega = \mathbf{e}_1\mathbf{e}_2. \tag{5}$$

In accordance with these correlations may be written (4) as

$$\mathcal{A}_0\mathcal{A}_1 = F = (E_1 + \omega H_1)\mathbf{e}_1 + (E_2 + \omega H_2)\mathbf{e}_2 + (E_3 + \omega H_3)\mathbf{e}_3 \tag{6}$$

It is obvious that the expression (6) is coincide with the vector part of complex quaternion (hyperbolic biquaternion) when $\mathbf{e}'_1 = i\mathbf{e}_1, \mathbf{e}'_2 = i\mathbf{e}_2, \mathbf{e}'_3 = i\mathbf{e}_1 i\mathbf{e}_2$. Moreover, by general definition, in the case of n is odd the element $\omega = \mathbf{e}_{12\ldots n}$ is belong to a center of $\mathbf{R}_n$, and when $n = 4m' - 1$ $(m' = 1, 2, \ldots)$

by force of $\omega^2=-1$ there is the identity $\omega=i$, where i is imaginary unit. Hence it follows that

$$\mathbf{R}_{4m'-1}=\mathbf{C}_{4m'-2}.$$

In particular case, when $m'=1$ we obtain $\mathbf{R}_3=\mathbf{C}_2$. Thus, in accordance with (4) and (6) we have the algebra $\mathbf{C}_2$ with general element $\mathcal{A}=F_0\mathbf{e}_0+F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3$, where $F_0=\partial_0A_0+\mathrm{div}\mathbf{A}\equiv 0$ and F_i $(i=1,2,3)-$ the components of complex electromagnetic field.

Further on, make up the exterior product $\bigtriangledown\mathbf{F}$, where $\bigtriangledown$ is the first element from (3) and $\mathbf{F}$ is an expression of type (6):

$$\bigtriangledown\mathbf{F}=\mathrm{div}\mathbf{E}\mathbf{e}_0-((\mathrm{curl}\mathbf{H})_1-\partial_0E_1)\mathbf{e}_1-((\mathrm{curl}\mathbf{H})_2-\partial_0E_2)\mathbf{e}_2-$$
$$-((\mathrm{curl}\mathbf{H})_3-\partial_0E_3)\mathbf{e}_3+((\mathrm{curl}\mathbf{E})_1+\partial_0H_1)\mathbf{e}_2\mathbf{e}_3+((\mathrm{curl}\mathbf{E})_2+\partial_0H_2)\mathbf{e}_3\mathbf{e}_1+ \quad (7)$$
$$+((\mathrm{curl}\mathbf{E})_3+\partial_0H_3)\mathbf{e}_1\mathbf{e}_2+\mathrm{div}\mathbf{H}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

It is easily seen that the first coefficient of the product $\bigtriangledown\mathbf{F}$ be a left part of equation $\mathrm{div}\mathbf{E}=\varrho$. The following three coefficients are make up a left part of equation $\mathrm{curl}\mathbf{H}-\partial_0\mathbf{E}=j$, the other coefficients are make up the equations $\mathrm{curl}\mathbf{E}+\partial_0\mathbf{H}=0$ and $\mathrm{div}\mathbf{H}=0$, respectively.

In accordance with (5) this product may be rewritten as

$$\bigtriangledown\mathbf{F}=(\mathrm{div}\mathbf{E}+\omega\mathrm{div}\mathbf{H})\mathbf{e}_0+(-((\mathrm{curl}\mathbf{H})_1-\partial_0E_1)+\omega((\mathrm{curl}\mathbf{E})_1+\partial_0H_1))\mathbf{e}_1+$$
$$+(-((\mathrm{curl}\mathbf{H})_2-\partial_0E_2)+\omega((\mathrm{curl}\mathbf{E})_2+\partial_0H_2)\mathbf{e}_2+$$
$$+(-((\mathrm{curl}\mathbf{H})_3-\partial_0E_3)+\omega((\mathrm{curl}\mathbf{E})_3+\partial_0H_3))\mathbf{e}_3.$$

It is obvious that in spinor representation of algebra $\mathbf{R}_3$ by force of identity $\mathbf{R}_3=\mathbf{C}_2$ we have an isomorphism $\mathbf{R}_3\cong\mathrm{M}_2(\mathbf{C})$. At this isomorphism the units $\mathbf{e}_i$ $(i=0,1,2,3)$ of $\mathbf{R}_3$ are correspond to the basis matrices of full matrix algebra $\mathrm{M}_2(\mathbf{C})$:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \qquad (8)$$

Hence it immediately follows that in spinor representation the first element (3) in the base (8) has a form (1), and, analoguosly, for the vector-potential $\mathbf{A}$ we have:

$$(a_{\lambda\dot{\mu}})=\begin{bmatrix}a_{1\dot{1}} & a_{1\dot{2}}\\ a_{2\dot{1}} & a_{1\dot{2}}\end{bmatrix}=\begin{bmatrix}A_0+A_3 & A_1+iA_2\\ A_1-iA_2 & A_0-A_3\end{bmatrix}.$$

Thus, for the matrix of spin-tensor f^{ρ}_{λ} and Maxwell's equations we obtain the following expressions:

$$(f^{\rho}_{\lambda})=(\partial^{\rho\dot{\sigma}}a_{\lambda\dot{\sigma}})=\begin{bmatrix}\partial^{1\dot{1}} & \partial^{1\dot{2}}\\ \partial^{2\dot{1}} & \partial^{2\dot{2}}\end{bmatrix}\begin{bmatrix}a_{1\dot{1}} & a_{1\dot{2}}\\ a_{2\dot{1}} & a_{2\dot{2}}\end{bmatrix}=$$

$$\begin{bmatrix} \partial_0+\partial_3 & \partial_1+i\partial_2 \\ \partial_1-i\partial_2 & \partial_0-\partial_3 \end{bmatrix} \begin{bmatrix} A_0+A_3 & A_1+iA_2 \\ A_1-iA_2 & A_0-A_3 \end{bmatrix} = \begin{bmatrix} F_3 & F_1+iF_2 \\ F_1-iF_2 & -F_3 \end{bmatrix},$$

$$(s^{\rho\dot{\mu}}) = (\partial^{\lambda\dot{\mu}} f^{\rho}_{\lambda}) = \begin{bmatrix} \partial_0+\partial_3 & \partial_1+i\partial_2 \\ \partial_1-i\partial_2 & \partial_0-\partial_3 \end{bmatrix} \begin{bmatrix} F_3 & F_1+iF_2 \\ F_1-iF_2 & -F_3 \end{bmatrix} =$$

$$\left[\begin{array}{c|c} \begin{array}{c}\mathrm{div}\mathbf{E}-(\mathrm{curl}\mathbf{H})_3+\partial_0 E_3+ \\ +i(\mathrm{div}\mathbf{H}+(\mathrm{curl}\mathbf{E})_3+\partial_0 H_3)\end{array} & \begin{array}{c}-(\mathrm{curl}\mathbf{H})_1+\partial_0 E_1+(\mathrm{curl}\mathbf{E})_2+\partial_0 H_2+ \\ +i((\mathrm{curl}\mathbf{E})_1+\partial_0 H_1-(\mathrm{curl}\mathbf{H})_2+\partial_0 E_2)\end{array} \\ \hline \begin{array}{c}-(\mathrm{curl}\mathbf{H})_1+\partial_0 E_1+(\mathrm{curl}\mathbf{E})_2+\partial_0 H_2- \\ -i((\mathrm{curl}\mathbf{E})_1+\partial_0 H_1-(\mathrm{curl}\mathbf{H})_2+\partial_0 E_2)\end{array} & \begin{array}{c}\mathrm{div}\mathbf{E}+(\mathrm{curl}\mathbf{H})_3-\partial_0 E_3- \\ -i((\mathrm{curl}\mathbf{E})_3+\partial_0 H_3+\mathrm{div}\mathbf{H})\end{array} \end{array}\right].$$

A dual exterior product we obtain by force of identity $* = \omega$, where $*$ is operator of Hodge[3] and $\omega = \mathbf{e}_{12\ldots n}$ is volume element of Clifford algebra $\mathbf{R}_n$. This identity is possible only if $n = 4m'+1$ or $n = 4m'-1$, where $m' = 1, 2, \ldots$; since only in this case ω is belong to a center of $\mathbf{R}_n$. For $\mathbf{R}_3$ we have $\omega = \mathbf{e}_{123}$. This way

$$\begin{aligned} *(\mathcal{A}_0\mathcal{A}_1) &= \omega(\mathcal{A}_0\mathcal{A}_1) = -H^1\mathbf{e}_0\mathbf{e}_1 - H^2\mathbf{e}_0\mathbf{e}_2 - H^3\mathbf{e}_0\mathbf{e}_3 + \\ &\quad +E^1\mathbf{e}_2\mathbf{e}_3 + E^2\mathbf{e}_3\mathbf{e}_1 + E^3\mathbf{e}_1\mathbf{e}_2. \end{aligned} \tag{9}$$

In the base (8) for a complex conjugate electromagnetic field from $*(i\mathcal{A}_0\mathcal{A}_1)$ we have:

$$(f^{\dot{\rho}}_{\dot{\lambda}}) = \begin{bmatrix} f^{\dot{1}}_{\dot{1}} & f^{\dot{1}}_{\dot{2}} \\ f^{\dot{2}}_{\dot{1}} & f^{\dot{2}}_{\dot{2}} \end{bmatrix} = \begin{bmatrix} \overset{*}{F}_3 & \overset{*}{F}_1 - i\overset{*}{F}_2 \\ \overset{*}{F}_1 + i\overset{*}{F}_2 & -\overset{*}{F}_3 \end{bmatrix},$$

where $\overset{*}{\mathbf{F}} = \mathbf{E} - i\mathbf{H}$.

Accordingly, a system of complex conjugate Maxwell's equations may be written as

$$\begin{aligned} \partial^{\mu\dot{\lambda}} f^{\dot{\rho}}_{\dot{\lambda}} &= 0, \\ \partial^{\mu\dot{\lambda}} f^{\dot{\rho}}_{\dot{\lambda}} &= s^{\mu\dot{\rho}}. \end{aligned}$$

In the terms of $\mathbf{R}_3 = \mathbf{C}_2$ this system is equivalent to a system of coefficients of exterior product $\bigtriangledown\overset{*}{F}$, where $\overset{*}{F}$ is a dual exterior product of type (9).

It is easily verified that a stress-energy tensor of electromagnetic field in spinor form is realized by spin-tensor $t^{\rho\dot{\mu}}_{\lambda\dot{\nu}} = f^{\rho}_{\lambda} f^{\dot{\mu}}_{\dot{\nu}}$, the matrix of which has a form:

$$(t^{\rho\dot{\mu}}_{\lambda\dot{\nu}}) = (f^{\rho}_{\lambda} f^{\dot{\mu}}_{\dot{\nu}}) = \begin{bmatrix} F_3 & F_1+iF_2 \\ F_1-iF_2 & -F_3 \end{bmatrix} \begin{bmatrix} \overset{*}{F}_3 & \overset{*}{F}_1 - i\overset{*}{F}_2 \\ \overset{*}{F}_1 + i\overset{*}{F}_2 & -\overset{*}{F}_3 \end{bmatrix} =$$

$$\left[\begin{array}{c|c} W-2s_3+i(\sigma_{21}-\sigma_{12}) & \sigma_{31}-2s_1-\sigma_{13}+i(\sigma_{32}-2s_2-\sigma_{23}) \\ \hline \sigma_{13}-2s_1-\sigma_{31}+i(\sigma_{32}+2s_2-\sigma_{23}) & W+2s_3+i(\sigma_{12}-\sigma_{21}) \end{array}\right],$$

where W is density of energy, σ_{ik} is stress tensor, and $\mathbf{s}$ is the vector of Poynting.

Thus, *we have a full description of the basic notions of electromagnetic field in the terms of algebra* $\mathbf{R}_3 = \mathbf{C}_2$ *and its spinor representation.*

Consider now an algebra $\mathbf{C}_4$. In the spinor representation this algebra is isomorphic to a matrix algebra $\mathrm{M}_4(\mathbf{C})$, the base of which consist of well-known Dirac γ-matrices. In the base of Weyl for these matrices we have

$$\gamma^m = \begin{bmatrix} 0 & \sigma^m \\ \overline{\sigma}^m & 0 \end{bmatrix},$$

where $m = 0, 1, 2, 3$ and

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\overline{\sigma}^0 = \sigma^0,\ \ \overline{\sigma}^{1,2,3} = -\sigma^{1,2,3}.$$

In this base the system of Dirac's equations has a following form:

$$\begin{bmatrix} 0 & 0 & \partial^{1\dot{1}} & \partial^{2\dot{1}} \\ 0 & 0 & \partial^{1\dot{2}} & \partial^{2\dot{2}} \\ \partial_{1\dot{1}} & \partial_{1\dot{2}} & 0 & 0 \\ \partial_{2\dot{1}} & \partial_{2\dot{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix} = -im \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix}.$$

Replace the symmetric spin-tensors $\partial^{\mu\dot{\lambda}}$ and $\partial_{\lambda\dot{\mu}}$ by the symmetric spin-tensors $s^{\mu\dot{\lambda}}$ and $s_{\lambda\dot{\mu}}$. Then

$$\left[\begin{array}{c|c} 0 & s^{\mu\dot{\lambda}} \\ \hline s_{\lambda\dot{\mu}} & 0 \end{array}\right] \begin{bmatrix} \xi \\ \dot{\eta} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \partial^{1\dot{1}} f^{\dot{1}}_{\dot{1}} + \partial^{2\dot{1}} f^{\dot{2}}_{\dot{1}} & \partial^{1\dot{1}} f^{\dot{1}}_{\dot{2}} + \partial^{2\dot{1}} f^{\dot{2}}_{\dot{2}} \\ 0 & 0 & \partial^{1\dot{2}} f^{\dot{1}}_{\dot{1}} + \partial^{2\dot{2}} f^{\dot{2}}_{\dot{1}} & \partial^{1\dot{2}} f^{\dot{1}}_{\dot{2}} + \partial^{2\dot{2}} f^{\dot{2}}_{\dot{2}} \\ \partial_{1\dot{1}} f^1_1 + \partial_{1\dot{2}} f^2_1 & \partial_{1\dot{1}} f^1_2 + \partial_{1\dot{2}} f^2_2 & 0 & 0 \\ \partial_{2\dot{1}} f^1_1 + \partial_{2\dot{2}} f^2_1 & \partial_{2\dot{1}} f^1_2 + \partial_{2\dot{2}} f^2_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix} = -im \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix}.$$

Hence we obtain the following system of equations:

$$\begin{aligned} (\partial^{1\dot{1}} f^{\dot{1}}_{\dot{1}} + \partial^{2\dot{1}} f^{\dot{2}}_{\dot{1}})\eta_{\dot{1}} + (\partial^{1\dot{1}} f^{\dot{1}}_{\dot{2}} + \partial^{2\dot{1}} f^{\dot{2}}_{\dot{2}})\eta_{\dot{2}} &= -im\xi^1, \\ (\partial^{1\dot{2}} f^{\dot{1}}_{\dot{1}} + \partial^{2\dot{2}} f^{\dot{2}}_{\dot{1}})\eta_{\dot{1}} + (\partial^{1\dot{2}} f^{\dot{1}}_{\dot{2}} + \partial^{2\dot{2}} f^{\dot{2}}_{\dot{2}})\eta_{\dot{2}} &= -im\xi^2, \\ (\partial_{1\dot{1}} f^1_1 + \partial_{1\dot{2}} f^2_1)\xi^1 + (\partial_{1\dot{1}} f^1_2 + \partial_{1\dot{2}} f^2_2)\xi^2 &= -im\eta_{\dot{1}}, \\ (\partial_{2\dot{1}} f^1_1 + \partial_{2\dot{2}} f^2_1)\xi^1 + (\partial_{2\dot{1}} f^1_2 + \partial_{2\dot{2}} f^2_2)\xi^2 &= -im\eta_{\dot{2}}. \end{aligned} \tag{10}$$

In particular case, when $f_1^1 = f_2^2 = f_{\dot{1}}^{\dot{1}} = f_{\dot{2}}^{\dot{2}} = 1$ and $f_1^2 = f_2^1 = f_{\dot{1}}^{\dot{2}} = f_{\dot{2}}^{\dot{1}} = 0$ system (10) is coincide with Dirac's equations

$$\begin{array}{rcl} \partial^{1\dot{1}}\eta_{\dot{1}} + \partial^{2\dot{1}}\eta_{\dot{2}} & = & -im\xi^1, \\ \partial^{1\dot{2}}\eta_{\dot{1}} + \partial^{2\dot{2}}\eta_{\dot{2}} & = & -im\xi^2, \\ \partial_{1\dot{1}}\xi^1 + \partial_{1\dot{2}}\xi^2 & = & -im\eta_{\dot{1}}, \\ \partial_{2\dot{1}}\xi^1 + \partial_{2\dot{2}}\xi^2 & = & -im\eta_{\dot{2}}. \end{array}$$

Analoguosly, when $\eta_{\dot{1}} = \xi^2 = 1,\ \eta_{\dot{2}} = \xi^1 = 0$ and $m = 0$ from (10) we obtain the following system of equations

$$\begin{array}{rcl} \partial^{1\dot{1}} f_{\dot{1}}^{\dot{1}} + \partial^{2\dot{1}} f_{\dot{1}}^{\dot{2}} & = & 0, \\ \partial^{1\dot{2}} f_{\dot{1}}^{\dot{1}} + \partial^{2\dot{2}} f_{\dot{1}}^{\dot{2}} & = & 0, \\ \partial_{1\dot{1}} f_2^1 + \partial_{1\dot{2}} f_2^2 & = & 0, \\ \partial_{2\dot{1}} f_2^1 + \partial_{2\dot{2}} f_2^2 & = & 0. \end{array}$$

or

$$\begin{array}{rcl} (\partial_0 + \partial_3)\overset{*}{F}_3 + (\partial_1 - i\partial_2)(\overset{*}{F}_1 + i\overset{*}{F}_2) & = & 0, \\ (\partial_1 + i\partial_2)\overset{*}{F}_3 + (\partial_0 - \partial_3)(\overset{*}{F}_1 + i\overset{*}{F}_2) & = & 0, \\ (\partial_0 + \partial_3)(F_1 + iF_2) - (\partial_1 + i\partial_2)F_3 & = & 0, \\ (\partial_1 - i\partial_2)(F_1 + iF_2) - (\partial_0 - \partial_3)F_3 & = & 0. \end{array} \tag{11}$$

Substitute in (11) $\overset{*}{\mathbf{F}} = \mathbf{E} - i\mathbf{H}$, $\mathbf{F} = \mathbf{E} + i\mathbf{H}$ and divide the real and imaginary parts after simple transformations we have

$$\begin{array}{rl} & \mathrm{div}\mathbf{E} - (\mathrm{curl}\mathbf{H})_3 + \partial_0 E_3 = 0, \\ - & \mathrm{div}\mathbf{H} - (\mathrm{curl}\mathbf{E})_3 - \partial_0 H_3 = 0, \\ - & (\mathrm{curl}\mathbf{H})_1 + \partial_0 E_1 + (\mathrm{curl}\mathbf{E})_2 + \partial_0 H_2 = 0, \\ - & (\mathrm{curl}\mathbf{H})_2 + \partial_0 E_2 - (\mathrm{curl}\mathbf{E})_1 - \partial_0 H_1 = 0, \\ - & (\mathrm{curl}\mathbf{H})_1 + \partial_0 E_1 - (\mathrm{curl}\mathbf{E})_2 - \partial_0 H_2 = 0, \\ - & (\mathrm{curl}\mathbf{H})_2 + \partial_0 E_2 + (\mathrm{curl}\mathbf{E})_1 + \partial_0 H_1 = 0, \\ & \mathrm{div}\mathbf{E} + (\mathrm{curl}\mathbf{H})_3 - \partial_0 E_3 = 0, \\ & \mathrm{div}\mathbf{H} - (\mathrm{curl}\mathbf{E})_3 - \partial_0 H_3 = 0. \end{array}$$

From the latest equations by means of addition and substraction we obtain the system of Maxwell's equations for empty space:

$$\begin{array}{l} \mathrm{div}\mathbf{E} = 0, \\ \mathrm{div}\mathbf{H} = 0, \\ \mathrm{curl}\mathbf{H} - \partial_0\mathbf{E} = 0, \\ \mathrm{curl}\mathbf{E} + \partial_0\mathbf{H} = 0. \end{array}$$

It is obvious that we obtain the same result if suppose in system (10) $m = 0$ and $\eta_{\dot{2}} = \xi^1 = 1,\ \eta_{\dot{1}} = \xi^2 = 0$.

Thus, *system* (10) *which we shall call the system of electron field equations, in the particular cases is coincide with Dirac's and Maxwell's equations.*

On the other hand, by force of isomorphism[4] $\mathbf{C}_4 \cong \mathbf{C}_2 \otimes \mathbf{C}_2$ or $\mathbf{C}_4 \cong \mathbf{C}_2 \otimes \overset{*}{\mathbf{C}}_2$, where $\overset{*}{\mathbf{C}}_2$ is an algebra with general element $\overset{*}{\mathcal{A}} = \overset{*}{F}_0\ \mathbf{e}_0 + \overset{*}{F}_1\ \mathbf{e}_1 + \overset{*}{F}_2\ \mathbf{e}_2 + \overset{*}{F}_3\ \mathbf{e}_{12}$, we may say that *the electron field be a tensor product of two photon fields.* Moreover, the algebras $\mathbf{C}_2$ and $\overset{*}{\mathbf{C}}_2$ are represent the photon fields with left-handed and right-handed polarization, respectively (see [5]).

References

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